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Control of Nonlinear Variable Structure Systems

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0. INTRODUCTION

We consider control systems described by *state equations*

$$\dot{x} = f(t, x, u), \tag{1}$$

where the state variable $x = (x_1, ..., x_n)' \in R^n$, the control variable $u = (u_1, ..., u_m)' \in R^m$ and a prime denotes transpose; *sliding manifold*

$$s(x) = (s_1(x), ..., s_m(x))' = 0 \tag{2}$$

and *control constraints* given by

$$u(t, x) \in \mathscr{U} \tag{3}$$

for some given subset $\mathscr{U}$ of R^m.

We wish to control the system by using feedback control laws $u = u(t, x)$ which are discontinuous along the surfaces given by

$$s_j(x) = 0, \qquad j = 1, ..., m.$$

Often the control law takes the form

$$u_j(t, x) = \begin{cases} u_j^+(t, x) & \text{if} \quad s_j(x) > 0 \\ u_j^-(t, x) & \text{if} \quad s_j(x) < 0, \end{cases} \tag{4}$$

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where u_j^+, u_j^-, $j = 1,\dots, m$, are given smooth functions verifying the constraint (3).

The performances of such control systems employing discontinuous feedback controls have been studied both from a theoretical and practical viewpoint. If we are able to keep the state vector, during a given time interval, on some suitably chosen sliding manifold (2) by using such controls, then we may get stable behavior, accurate tracking, robust performance, insensitivity with respect to disturbances, and variations of plant parameters, at least in principle. It is possible to show that by the introduction of a discontinuous control law, control systems are obtained characterized by performances which either cannot be achieved by continuous control or can only be achieved by employing more complex structures. About these features we refer the reader to [4] and [1], where a systematic approach has been developed to the analysis of variable structure systems. See [3] for a discussion of some drawbacks of these techniques, of modifications required to alleviate them and of some applications to robotics.

The mathematical theory of discontinuous control systems (1), (2), (3), (4) has been developed so far mainly when

$$\mathscr{U} = R^m, f(t, x, u) = A(t, x) + B(t, x) u \tag{5}$$

(see [1]). It is based on the Filippov definition of solution for ordinary differential equations with discontinuous right-hand side (see [2]). Related results may be found in [16]: however, nonlinear discontinuous control systems require further analysis concerning their control theoretical behaviour, as we shall see in Section 2. When f depends affinely upon the control variables as in (5), a suitable definition of equivalent control has been introduced in [1]. This allows us to get, by a continuous control law, the same states as given by the Filippov definition whenever we consider sliding modes of the system, i.e., states x such that $s[x(t)] = 0$ in some time interval. As shown in [1, p. 45], under suitable conditions the equivalent control notion has a physical meaning under assumption (5). As a matter of fact, the real states of the systems converge to the ideal states generated by the equivalent control as the perturbations acting on the systems disappear (such perturbations may include delays, hysteresis, small time constants, errors in the control law, approximate description of the plant based on incomplete knowledge).

In this paper we extend the theory of variable structure discontinuous control systems to some nonlinear cases, weakening assumption (5).

As remarked in [3, p. 489, Sect. 7] the sliding-mode control methodology needs to be extended to more general classes of nonlinear systems than those considered so far. Such a need is mentioned also in [1, p. 68].

In Section 1 we introduce for some nonlinear control systems a suitable definition of equivalent control, thereby generalizing the definition of [1] which required assumption (5). With suitable assumptions we prove the equivalence between sliding modes generated by our equivalent control and Filippov solutions of (1) corresponding to discontinuous feedback control laws. Within our definition independence is obtained, up to a large extent, of system performances upon the choice of the discontinuous feedback law (4) as far as we keep the state variable on the sliding manifold (2).

In Section 2 we introduce a definition of approximability for nonlinear control systems. In this way we are able to extend to the fully nonlinear setting a physically relevant property, discussed in [1] under assumption (5). By considering some examples we show that not every nonlinear system, for which the equivalent control exists, satisfies such a property, thereby validating in a rigorous way a conjecture stated in [1] (see also [4, p. 218]) and based on an example, heuristically discussed in [1, p. 64]. By using our definition of approximability we are able to treat rigorously such an example. Thus for some nonlinear systems the concept of equivalent control may be lacking of physical meaning.

The approximability property we define in this paper is new, and related to the G-convergence of ordinary differential equations (see [5]).

At the end of the paper we show that the equivalent control may be defined and the approximability property is verified for each of the following nonlinear control systems:

(i) $z^{(n)} = g(t, z, z', z'', ..., z^{(n-1)}, u)$

with z and u scalar variables;

(ii) $\dot{x} = A(t, x) + B(t, x)\, h(u)$,

x and u vector variables; both under explicitly verifiable conditions about the data.

We refer to [13] for a discussion of some of these results from the viewpoint of their control engineering applications. Further results and applications will be considered elsewhere.

1. Equivalent Control and Filippov Solutions

We shall consider the control system (1), (2), (3) using sometimes the following conditions (6), (7).

f is a Carathéodory function from $[0, T] \times \Omega \times \mathscr{U}$ to R^n for a given $T > 0$ and some open $\Omega \subset R^n$. (6)

We shall assume throughout the paper that $s \in C^1(\Omega)$. Think of Ω as an open set containing every instantaneous state of the given system. Let us denote by

$$G = \frac{\partial s}{\partial x}$$

the $m \times n$ jacobian matrix of elements $\partial s_i/\partial x_j$, $i = 1,..., m$; $j = 1,..., n$, and by S the sliding manifold given by (2), so that

$$S = \{x \in R^n : s_j(x) = 0, j = 1,..., m\}.$$

We shall use the following condition.

$$\text{There exists a neighborhood } V \text{ of } S \text{ such that for every } (t, x) \in [0, T] \times V \text{ the map } G(x) f(t, x, \cdot) \text{ is one-to-one on } \mathscr{U} \text{ and its range contains } 0. \tag{7}$$

The unique solution (if any) $u \in \mathscr{U}$ of the equation

$$G(x) f(t, x, u) = w$$

for a given $w \in R^m$ will be denoted by

$$u^*(t, x, w).$$

We refer the reader to [8] for a survey with many explicit sufficient conditions for global injectivity of $G(x) f(t, x, \cdot)$.

DEFINITION 1. Assume conditions (6) and (7). The *equivalent control* for the system (1), (2), (3) is the mapping

$$(t, x) \to u^*(t, x, 0),$$

$$0 \leqslant t \leqslant T,\ x \in V \qquad \text{some neighborhood of } S.$$

The above definition generalizes that given in [1] under assumption (5), since in that case condition (7) amounts to

$$\det G(x)\, B(t, x) \neq 0,$$

i.e., the nonsingularity condition introduced in [1, p. 44].

By using discontinuous feedback control laws as in (4) we need a concept of solution of (1) under nonclassical conditions. The relevant definition used here is that of Filippov (see [2]). Although different notions of solution could be considered here, the results of [1, 2, 6] show clearly that the Filippov definition is significant for many control systems, including those satisfying (5). For different approaches see [7, 17].

Let us recall the definition introduced by Filippov in [2]. Let g a R^n-valued Lebesgue measurable function defined a.e. on $[0, T] \times \Omega$. We denote by ch the closure, by co the convex envelope, by $B(x, \delta)$ the open ball of radius δ around $x \in R^n$. Then y is a *Filippov solution in* $[0, T]$ of the ordinary differential system

$$\dot{x} = g(t, x)$$

if y is absolutely continuous in $[0, T]$ and for a.e. $t \in [0, T]$

$$\dot{y}(t) \in \text{ch co } g[t, B(y(t), \delta) \setminus N]$$

for every $\delta > 0$ and every set N of zero Lebesgue n-dimensional measure. We shall say that y is an *a.e. solution in* $[0, T]$ of $\dot{x} = g(t, x)$ iff y is absolutely continuous in $[0, T]$ and for a.e. $t \in [0, T]$.

$$\dot{y}(t) = g[t, y(t)].$$

Generalizing results obtained in [1] under assumption (5) we shall show that the state trajectories on the sliding manifold (2) corresponding to many discontinuous feedback control laws may be obtained as a.e. solutions of (1) given by the equivalent control, and conversely, if suitable assumptions hold about (1), (2), (3). Such results will justify definition 1. Moreover solving (1) in the sense of Filippov can be avoided by using the equivalent control. The next theorems will show that the dynamics of the state trajectory on the sliding manifold are completely specified by the constraints (2) to stay on it for many nonlinear control systems. Thus these dynamics are insensitive to parameter variations, disturbances and to the particular feedback control law employed.

We shall use the following condition:

$$m \leqslant n \quad \text{and} \quad \text{rank } G(x) = m \quad \text{for all} \quad x \in S. \tag{8}$$

If (8) holds, we can find a positive integer p such that for every $x_0 \in S$ and some $\delta > 0$ the ball of center x_0, radius δ, may be written as disjoint union of subsets of the surfaces

$$S_j = \{x \in R^n : s_j(x) = 0\}, \qquad j = 1, \dots, m$$

and of p open connected regions $C_1, \dots, C_p$.

We shall adhere to the following terminology. The function

$$v \colon [0, T] \times \left(\Omega \setminus \bigcup_{j=1}^{m} S_j \right) \to R^r$$

is *locally integrably bounded* iff for every compact $K \subset \Omega \setminus \bigcup_{j=1}^{m} S_j$ there exists $w \in L^1(0, T)$ such that

$$|v(t, x)| \leqslant w(t)$$

for a.e. $t \in [0, T]$ and every $x \in K$. The function $q = q(t, x, u)$ is *locally integrably bounded near* S iff there exists a neighborhood V of S such that given any compact $K \subset V$ we can find $w \in L^1(0, T)$ so that

$$|q(t, x, u)| \leqslant w(t)$$

for a.e. $t \in [0, T]$, all $x \in K$ and $u \in \mathscr{U}$. If w above can be chosen constant, we say that q is *locally bounded near* S.

We shall denote by x $([0, T])$ the set of values $x(t)$ as $0 \leqslant t \leqslant T$.

LEMMA 1. *Let* $g\colon [0, T] \times (\Omega \setminus \bigcup_{j=1}^{m} S_j) \to R^n$ *be measurable in* t, *continuous in* x, *locally integrably bounded. Let us assume* (8) *and let* x *be absolutely continuous on* $[0, T]$ *such that* $s[x(t)] = 0$ *there. Let* g^j *be the restriction of* g *to the region* C_j, *defined above. Assume that for every* $x_0 \in x([0, T])$ *there exists a finite*

$$g^j(t, x_0) = \lim_{x \to x_0} g^j(t, x), \qquad j = 1, \ldots, p.$$

Then if x *is a Filippov solution in* $[0, T]$ *of*

$$\dot{x} = g(t, x),$$

for a.e. t *we get*

$$\dot{x}(t) \in \operatorname{co}\{ g^j[t, x(t)] \colon j = 1, \ldots, p\}.$$

Proof. For a.e. $t \in [0, T]$

$$\dot{x}(t) \in \bigcap_{\delta > 0} \cap \{\operatorname{ch} \operatorname{co} g(t, B[x(t), \delta] \setminus N) \colon \operatorname{meas} N = 0\}. \tag{9}$$

By Lemma 1, p. 203 of [2], given t such that (9) holds, for every $\delta > 0$ there exists $M \subset R^n$, meas $M = 0$, such that

$$\dot{x}(t) \in \operatorname{ch} \operatorname{co} g(t, B[x(t), \delta] \setminus M).$$

By taking $\delta = 1/k$, $k = 1, 2, 3, \ldots$, we see that $\dot{x}(t)$ is limit as $k \to +\infty$ of points

$$q_k = \sum_j \alpha^1_{jk}\, g^1(t, x^1_{jk}) + \sum_j \alpha^2_{jk}\, g^2(t, x^2_{jk}) + \cdots + \sum_j \alpha^p_{jk}\, g^p(t, x^p_{jk}),$$

where the terms above are at most $n+1$ by Carathéodory's theorem 17.1 of [9], $\alpha^r_{jk} \geqslant 0$,

$$\sum_j \alpha^1_{jk} + \cdots + \sum_j \alpha^p_{jk} = 1,$$

$$x^r_{jk} \in C_r, \qquad |x^r_{jk} - x(t)| < \frac{1}{k}, \qquad r = 1, \ldots, p$$

and every k sufficiently large. Taking subsequences we assume α^r_{jk} converging as $k \to +\infty$. Thus by continuity of g and convergence of g^j on S, $j = 1, \ldots, p$, we can find numbers $\alpha_j \geqslant 0$ which sum up to one, such that

$$\dot{x}(t) = \sum_{j=1}^{p} \alpha_j\, g^j(t, x(t)). \qquad \text{Q.E.D.}$$

Lemma 1 is an extension of Lemma 3, p. 206 in [2].

In the following results we shall denote by Q the set of all Carathéodory feedback control laws

$$u = u(t, x): \quad [0, T] \times \left(\Omega \setminus \bigcup_{j=1}^{m} S_j \right) \to \mathscr{U}$$

such that $f[t, x, u(t, x)]$ is locally integrably bounded, and for every $x_0 \in S$ there exists a finite limit

$$u^j(t, x_0) = \lim_{x \to x_0} u^j(t, x), \quad j = 1, \ldots, p,$$

where u^j denotes the restriction of u to C_j.

THEOREM 1. *Assume conditions* (6), (7), (8). *Let* $u \in Q$. *Let* y *be a Filippov solution on* $[0, T]$ *of*

$$\dot{x} = f[t, x, u(t, x)]$$

such that $s[y(t)] = 0$, $0 \leqslant t \leqslant T$. *If*

$$\mathscr{U} \text{ is closed,} \tag{10}$$

$$f(t, x, \mathscr{U}) \text{ is convex for every } t \in [0, T] \text{ and } x \in y([0, T]) \tag{11}$$

then y is a.e. solution of (1) *on* $[0, T]$ *corresponding to the equivalent control.*

Proof. Given $x_0 \in y([0, T])$, by continuity of $f(t, \cdot, \cdot)$ we get convergence as $x \to x_0$ of

$$g^j(t, x) = f[t, x, u^j(t, x)], \qquad j = 1, \dots, p.$$

Thus $u^j(t, x_0) \in \mathscr{U}$ by (10) while Lemma 1 implies for a.e. t

$$\dot{y}(t) \in \operatorname{co}\{f[t, y(t), u^j(t, y(t))]: j = 1, \dots, p\}.$$

From (11) we can find $v(t) \in \mathscr{U}$ such that for a.e. t

$$\dot{y}(t) = f(t, y(t), v(t)), \tag{12}$$

then for such t

$$G[y(t)]\,\dot{y}(t) = \frac{d}{dt}\, s[y(t)] = 0$$
$$= G[y(t)]\, f[t, y(t), v(t)].$$

Since $G(x) f(t, x, \cdot)$ is one-to-one it follows

$$v(t) = u^*(t, y(t), 0)$$

and (12) gives the conclusion. Q.E.D.

In the next theorem we shall denote by $\partial G/\partial x\, f$ the $m \times n$ matrix of elements

$$\sum_{k=1}^{n} f_k \frac{\partial^2 s_j}{\partial x_k\, \partial x_r}, \qquad j = 1, \dots, m \text{ and } r = 1, \dots, n,$$

and by Q_0 the set of those $u \in Q$ such that if x is any Filippov solution to

$$\dot{x} = f(t, x, u(t, x)), \qquad x(0) \in S$$

then $x(t) \in S$ if $0 \leqslant t \leqslant T$.

THEOREM 2. *Let y be an a.e. solution of* (1) *in* $[0, T]$ *corresponding to the equivalent control, such that* $s[y(0)]=0$. *Assume conditions* (6), (7), (8), (10), $s \in C^2(\Omega)$ *and suppose that*

$$f(t, x, \mathscr{U}) \qquad \text{is convex if } 0 \leqslant t \leqslant T,\ x \in S; \tag{13}$$

f is continuously differentiable with respect to (x, u); $G(x)(\partial f/\partial u)(t, x, u)$ is nonsingular whenever $0 \leqslant t \leqslant T$, x is near S and $u \in \mathscr{U}$; (14)

$\partial f/\partial u$ is locally bounded near S, and $\partial f/\partial x$, af_j, $a(\partial f_i/\partial x_j)$ are locally integrably bounded near S for every i, j and every element a of $(G(\partial f/\partial u))^{-1}$. (15)

Then y is a Filippov solution in $[0, T]$ *of*

$$\dot{x}=f[t, x, u(t, x)]$$

for every feedback control law $u \in Q_0$.

Proof. By the assumptions $(t, x) \rightarrow f[t, x, u(t, x)]$ is measurable and locally integrably bounded. By Theorem 4, p. 212 of [2] there exist Filippov solutions of

$$\dot{x}=f[t, x, u(t, x)], \qquad x(0)=y(0)$$

at least locally. Let z be any of them, defined if $0 \leqslant t \leqslant T$ with $z(t) \in S$ for every t. By theorem 1 we see that y and z are a.e. solutions in $[0, T]$ of

$$\dot{x}=f[t, x, u^*(t, x, 0)], \qquad x(0)=y(0), \quad x(t) \in S \text{ if } 0 \leqslant t \leqslant T. \tag{16}$$

We apply the implicit function theorem to

$$G(x) f(t, x, u)=0.$$

Remembering (14) we get for all $t \in [0, T]$ and x near S

$$\frac{\partial G}{\partial x} f+G \frac{\partial f}{\partial x}+G \frac{\partial f}{\partial u} \frac{\partial u^*}{\partial x}=0,$$

where $u^*=u^*(t, x, 0)$. By (15) we see that $\partial u^*/\partial x$ is locally integrably bounded along with partial derivatives of the components of $f[t, x, u^*(t, x, 0)]$ with respect to x_i. Thus (16) has uniqueness, so that $y=z$. Q.E.D.

Remark 1. If in Theorem 2 we assume

$$|f[t, x, u(t, x)]| \leqslant w(t), \qquad 0 \leqslant t \leqslant T,\ x \in V$$

and V contains the ball $B[y(0), \int_0^T w(t)\,dt]$, then by [2, Theorem 4, p. 212] we get Filippov solutions in the large.

Remark 2. Assumptions (14), (15) may be varied without altering the conclusions of Theorem 2. For example, we may assume that af_j, $a(\partial f_j/\partial x_i)$, and $|\partial f/\partial u|$ are bounded by square summable functions. More generally it suffices to assume local Lipschitz continuity of f with respect to x and u and conditions giving the local Lipschitz behavior of u^* (by using the results of [10, Sect. 7.1]).

As a particular case assume that the control variable is scalar and f depends affinely on it, that is,

$$f(t, x, u) = A(t, x) + B(t, x)\,u; \qquad m = 1. \tag{17}$$

In this case we have only one sliding surface

$$s(x) = 0.$$

Then the equivalence between states corresponding to the equivalent control and Filippov solutions of (1) in the sliding mode may be directly obtained in a simpler way. The reason is the following. If the control variable is scalar and enters linearly in the dynamics, the equivalent control may be characterized as a convex mean depending on an explicitly determined coefficient.

THEOREM 3. *Assume* (17) *and suppose* A, B *bounded Carathéodory functions, and* $\mathscr{U}$ *an interval. Let* F *be the set of all bounded feedback control laws* (4), *such that for every* $x_0 \in S$ *there exist finite limits*

$$u^{\pm}(t, x_0) = \lim_{x \to x_0} u^{\pm}(t, x), \qquad 0 \leqslant t \leqslant T,$$

such that if $x \in S$ *and* $0 \leqslant t \leqslant T$

$$GA + GBu^- \geqslant 0 \geqslant GA + GBu^+, \qquad GB(u^- - u^+) > 0;$$

$GB \neq 0$ *if* x *is near* S, *for every* t. *Then* y *is a.e. solution of* (1) *in* $[0, T]$ *corresponding to the equivalent control,* $y(0) \in S$, *iff* y *is a Filippov solution of* (1) *in* $[0, T]$ *corresponding to some feedback belonging to* F, $y(t) \in S$ *if* $0 \leqslant t \leqslant T$.

Proof. The equivalent control is given by

$$u^*(t, x, 0) = -(GB)^{-1}\,GA.$$

Fix $u \in F$ and consider

$$\alpha = (GA + GBu^-)[GB(u^- - u^+)]^{-1}.$$

Then $0 \leqslant \alpha \leqslant 1$, moreover

$$\alpha u^{+} + (1-\alpha) u^{-} = -(GB)^{-1} GA. \tag{18}$$

Let y be an a.e. solution of (1) given by the equivalent control. Then by (18) for every $u \in F$

$$\dot{y} = \alpha(A + Bu^{+}) + (1-\alpha)(A + Bu^{-})$$

for a.e. t. Moreover $y(t) \in S$ for all t, then y is a Filippov solution of (1) by Lemma 3, p. 206 of [2]. Conversely let y be a Filippov solution of (1) corresponding to some $u \in F$. Then by Lemma 3 of [2] and (18) we see that y is an a.e. solution of (1) given by the equivalent control. Q.E.D.

Theorem 3 extends previous results of [1]. The convexity assumption (13) is true, e.g., in each of the following cases:

(i) $f_j(t, x, u) = x_{j+1}$, $1 \leqslant j \leqslant n-1$,
$f_n(t, x, u) = g(t, x, u)$

for some real-valued Carathéodory function g, $m = 1$ and U is any interval;

(ii) $f(t, x, u) = A(t, x) + B(t, x)\, h(u)$,

where $h(U)$ is a convex set.

The following example shows that the convexity assumptions about $f(t, x, \mathscr{U})$ cannot be omitted in the above theorems.

EXAMPLE 1. Consider the control system

$$\dot{x}_1 = u, \qquad \dot{x}_2 = u^2, \qquad 2\,|u| \leqslant 1, \quad T = 1, \quad s(x_1, x_2) = x_1 - x_2.$$

Since $G(x)\, f(t, x, u) = u - u^2$ we have $u^*(t, x, 0) = 0$. Consider the piecewise constant feedback

$$2u^{+} = 1 \qquad \text{if} \quad x_1 > x_2,$$

$$2u^{-} = -1 \qquad \text{if} \quad x_1 < x_2.$$

Then $z(t) = \binom{t/4}{t/4}$ is a sliding mode generated in the sense of Filippov by the above feedback, since

$$4F(t, x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{if} \quad x_1 > x_2,$$

$$4F(t, x) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \text{if} \quad x_1 < x_2,$$

$$4F(t, x) = \left\{ \begin{pmatrix} 4\alpha - 2 \\ 1 \end{pmatrix} : 0 \leqslant \alpha \leqslant 1 \right\} \qquad \text{if} \quad x_1 = x_2,$$

where $F(t, x)$ is the right-hand side in (9) for this case. But any sliding mode x given by the equivalent control has $x_1(t) = x_2(t) =$ constant, and

$$\dot{x}(t) = 0 \notin F(t, x(t)), \qquad 0 \leqslant t \leqslant 1,$$

Remark. The results of Theorems 1 and 2 may be related to viability theory, for which we refer the reader to [14] and [15]. As a matter of fact, we may consider the differential inclusion

$$\dot{x}(t) \in F(t, x(t))$$

subject to the constraint (viability domain)

$$x(t) \in s^{-1}(0), \qquad 0 \leqslant t \leqslant T,$$

where

$$F(t, x) = f(t, x, U).$$

Under the assumptions of Theorems 1, 2, the Filippov solutions to (1) are a.e. solutions of the above differential inclusion. The notion of equivalent control reduces here to that of feedback map considered in [15]. On the other hand the (more particular) explicit description of control system (1), (3) used in this paper was chosen to define directly the relevant approximability property we need to consider, and to describe explicitly some classes of nonlinear control systems to which such a property applies (see the next section).

2. Approximability

In this section we introduce formally the following physically relevant property (see [1, especially pp. 30, 58, 64] for an interesting discussion of relevant examples).

The sliding modes of the control systems (1), (2), (3) may be uniformly approximated by states x of the system fulfilling only approximately the sliding condition $s[x(t)] = 0$ as the imperfections causing such a behavior disappear. Such states are called real in [1] while states realizing exactly the sliding conditions are called ideal.

Definition 2. Given $p > 1$, $M \in L^p(0, T)$ and a neighborhood V of S we denote by H the set of all generalized sequences of R^m-valued functions $a_\varepsilon \in L^p(0, T)$, $\varepsilon > 0$, such that

$$|a_\varepsilon(t)| \leqslant M(t), \ \varepsilon > 0$$

and a.e. $t \in [0, T]$,

$$\operatorname{Sup}\left\{\left|\int_0^t a_\varepsilon(s)\, ds\right| : 0 \leqslant t \leqslant T\right\} \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{19}$$

and such that there exists $u^*[t, x, a_\varepsilon(t)]$ for a.e. $t \in [0, T]$ and $x \in V$.

DEFINITION 3. System (1), (2), (3) fulfils the *approximability property* iff (7) holds and there exist $p > 1$, $M \in L^p(0, T)$, such that H (with V given by (7)) is non empty and for every

$a_\varepsilon \in H$, if x_ε, $\varepsilon > 0$, is a.e. solution in $[0, T]$ of

$$\dot{x} = f[t, x, u^*(t, x, a_\varepsilon(t)],\tag{20}$$

$s[x_\varepsilon(0)] \to 0$ as $\varepsilon \to 0$, if y is a.e. solution in $[0, T]$ of

$$\dot{x} = f[t, x, u^*(t, x, 0)],\tag{21}$$

$s[y(0)] = 0$, then

$x_\varepsilon(0) \to y(0)$ implies $x_\varepsilon \to y$ uniformly in $[0, T]$ as $\varepsilon \to 0$.

Discussion of Definition 3

Given $a_\varepsilon \in H$, x_ε a.e. solution of (20), put

$$u_\varepsilon(t) = u^*[t, x_\varepsilon(t), a_\varepsilon(t)].$$

Then for a.e. $t \in [0, T]$,

$$\frac{d}{dt} s[x_\varepsilon(t)] = G[x_\varepsilon(t)]\, f[t, x_\varepsilon(t), u_\varepsilon(t)]$$
$$= a_\varepsilon(t)$$

and by integrating between 0 and t, from (19) we get

$$s[x_\varepsilon(t)] \to 0 \qquad \text{uniformly on } [0, T] \text{ as } \varepsilon \to 0.$$

Then the approximability property holds for the control system (1), (2), (3) iff

(i) we can uniformly approximate any ideal sliding state by real states realizing only approximately the sliding condition, no matter of the disturbances causing the sliding error, which is measured by a_ε (and tends to zero in the Sobolev space $H^{-1,\infty}(0, T)$ as required in (19));

(ii) the real states of the system converge towards a well-defined ideal state whenever the initial values tend to the sliding manifold, as the disturbances disappear (in the sense (19)).

The boundedness by M required in Definition 2 does not restrict the range of applications and is needed here for technical reasons only.

Definition 3 generalizes the property informally introduced in [1, p. 44 and Theorem, p. 45], where the parameter ε measures the error in the real sliding with respect to the ideal one, through the condition

$$|s[x_\varepsilon(t)]| \leqslant \varepsilon.$$

Clearly in Definition 3 the exact nature of ε is immaterial: any metric space Q with a fixed element ε_0 and $\varepsilon \to \varepsilon_0$ in Q will do.

Remark. Definition 3 makes sense only when uniqueness holds for a.e. solutions of Cauchy problems for (21).

The following example was given in [1, p. 64] to support the following claim. For non linear control systems the sliding modes obtained in the limit with nonidealities tending to zero may be defined in a nonunique way. By using definition 3 we can put this example in a rigorous framework.

EXAMPLE 2. Consider the system

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = u_1 u_2; \quad T = 1;$$

$$|u_1| \leqslant 1, \qquad |u_2| \leqslant 1;$$

$$s_1(x) = x_1, \qquad s_2(x) = x_2.$$

Condition (7) is fulfilled since

$$G(x)\, f(t, x, u) = u.$$

Consider now $\varepsilon = 1/j$, $j = 1, 2, 3, \ldots$, and

$$v_j(t) = \begin{cases} 1 & \text{if } \; K/j \leqslant t < (2K+1)/2j \\ -1 & \text{if } \; (2K+1)/2j \leqslant t < (K+1)/j, \end{cases}$$

$0 \leqslant K \leqslant j - 1$. Of course

$$\left| \int_0^t v_j(s)\, ds \right| \leqslant \frac{1}{j}$$

so that (19) holds with

$$a_j(t) = \begin{pmatrix} v_j(t) \\ v_j(t) \end{pmatrix}.$$

Since $u^*(t, x, w) = w$, we consider

$$\begin{aligned} \dot{x}_{1j} &= v_j, & x_{1j}(0) &= 0; \\ \dot{x}_{2j} &= v_j, & x_{2j}(0) &= 0; \\ \dot{x}_{3j} &= v_j^2, & x_{3j}(0) &= 0; \\ \dot{y}_1 = \dot{y}_2 = \dot{y}_3 &= 0, & y_1(0) = y_2(0) = y_3(0) &= 0. \end{aligned}$$

Then $x_j \not\to y$ as $j \to +\infty$, contrary to Definition 3, since $x_{3j}(t) = t$ while $y_3(t) = 0$ for all t.

Let us remark in passing that in the example above condition (11) is not fulfilled.

The approximability property is related to solution convergence of initial value problems for ordinary differential systems, known as G-convergence. We refer the reader to [5] for a survey about such problems.

We shall need the following

LEMMA 2. *Let g_ε: $[0, T] \times \Omega \to R^n$ be Carathéodory functions, $\varepsilon \geqslant 0$, such that for every compact $K \subset \Omega$ there exists $C \in L^1(0, T)$ with*

$$|g_\varepsilon(t, x') - g_\varepsilon(t, x'')| \leqslant C(t)\, |x' - x''| \tag{22}$$

for a.e. t, $\varepsilon \geqslant 0$, x' and x'' in K; there exist A, $B \in L^P(0, T)$, $P > 1$ with

$$|g_\varepsilon(t, x)| \leqslant A(t) + B(t)\, |x| \tag{23}$$

for a.e. t, $\varepsilon \geqslant 0$, $x \in \Omega$.

Assume that for every $x \in \Omega$,

$$\operatorname{Sup} \left\{ \left| \int_0^t [g_\varepsilon(t, x) - g_0(t, x)]\, dt \right| : 0 \leqslant t \leqslant T \right\} \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{24}$$

If x_ε is a.e. solution on $[0, T]$ of

$$\dot{x} = g_\varepsilon(t, x), \qquad \varepsilon \geqslant 0,$$

then $x_\varepsilon(0) \to x_0(0)$ implies $x_\varepsilon \to x_0$ uniformly on $[0, T]$.

Proof. Uniqueness in the large holds for every initial value problem we are considering by (22) and (23). Let z_ε the a.e. solution in $[0, T]$ of

$$\dot{z} = g_\varepsilon(t, z),\ z(0) = x_0(0).$$

Then $z_\varepsilon \to x_0$ uniformly on $[0, T]$ by Theorem 1, p. 469 of [11] and the remarks thereof (p. 467 after Definition 2). Since x_ε, z_ε are uniformly bounded by (23) we get from (22)

$$|g_\varepsilon(s, x_\varepsilon(s)) - g_\varepsilon(s, z_\varepsilon(s))| \leqslant C(s)\, |x_\varepsilon(s) - z_\varepsilon(s)|$$

for some $C \in L^1(0, T)$, a.e. $s \in [0, T]$, $\varepsilon \geqslant 0$. Since

$$x_\varepsilon(t) - z_\varepsilon(t) = x_\varepsilon(0) - z_\varepsilon(0) + \int_0^t \{g_\varepsilon[s, x_\varepsilon(s)] - g_\varepsilon[s, z_\varepsilon(s)]\}\, ds$$

by Gronwall's lemma it follows $x_\varepsilon - z_\varepsilon \to 0$ uniformly on $[0, T]$ thus $x_\varepsilon \to x_0$ as required. Q.E.D.

Remark 1. The assumption $p > 1$ cannot be weakened to $p = 1$ in (23) as shown in [12]. This is the reason of assuming $p > 1$ in Definition 2.

Remark 2. As shown in Theorem 1 of [11], condition (24) is necessary (and sufficient) to get uniform convergence of solutions of any initial value problem for $\dot{x} = g_\varepsilon(t, x)$.

LEMMA 3. *Let a_ε, $\varepsilon > 0$, be bounded in $L^p(0, T)$, $p > 1$, and verify* (19). *If $z \in L^q(0, T)$, $(1/p) + (1/q) = 1$, then*

$$\int_0^t z(s)\, a_\varepsilon(s)\, ds \to 0 \textit{ uniformly on } [0, T] \textit{ as } \varepsilon \to 0.$$

Proof. Given $x \in C^1(0, T)$ consider

$$b_\varepsilon(t) = \int_0^t a_\varepsilon(s)\, ds.$$

Integrating by parts

$$\int_0^t x a_\varepsilon\, ds = x(t)\, b_\varepsilon(t) - \int_0^t \dot{x} b_\varepsilon\, ds.$$

Then for a suitable constant C

$$\left|\int_0^t x a_\varepsilon\, ds\right| \leqslant C \max\{|b_\varepsilon(t)| : 0 \leqslant t \leqslant T\} \to 0 \qquad \text{as } \varepsilon \to 0.$$

Thus the lemma is proved if $z \in C^1(0, T)$. A simple density argument completes the proof. Q.E.D.

We shall use the following terminology. The function $v = v(t, x)$, $0 \leqslant t \leqslant T$, $x \in \Omega$, is *locally p-integrably Lipschitz* iff for every compact $K \subset \Omega$ there exists $b \in L^p(0, T)$ such that for a.e. t and x', $x'' \in K$,

$$|v(t, x') - v(t, x'')| \leqslant b(t)\, |x' - x''|.$$

The function v is *of linear growth* if for some $p > 1$, c and r in $L^p(0, T)$ we have for all $x \in \Omega$ and a.e. t

$$|v(t, x)| \leqslant c(t) + r(t)\, |x|.$$

If necessary we shall say that v is *of linear p-growth.*

The next result shows that the approximability property holds if the control variables enter linearly in the dynamics of the system.

COROLLARY 1. *Let*

$$f(t, x, u) = A(t, x) + B(t, x)\, u,$$

A and B Carathéodory functions. Assume $G(x)\, B(t, x)$ nonsingular for every $x \in \Omega$ and a.e. t; *near S, $-[(GB)^{-1}\, GA](t, x) \in U$;*

$$B(\cdot, x)[G(x)\, B(\cdot, x)]^{-1} \in L^q(0, T), \qquad q > 1.$$

Suppose that $A - B\,(GB)^{-1}\, GA$ is locally 1*-integrably Lipschitz and of linear growth, $B(GB)^{-1}$ is locally q-integrably Lipschitz and of linear q-growth, $q > 1$. Then system* (1), (2), (3) *fulfils the approximability property.*

Proof. Given a suitable neighborhood V of S, let $p > q/(q-1)$, $M \in L^p(0, T)$ and $a_\varepsilon \in H$. Then

$$\begin{aligned} f[t, x, u^*(t, x, a_\varepsilon(t))] &- f[t, x, u^*(t, x, 0)] \\ &= B(t, x)[G(t, x)\, B(t, x)]^{-1}\, a_\varepsilon(t). \end{aligned}$$

In view of Lemma 2, if suffices to prove that for every $x \in \Omega$,

$$\int_0^t B(GB)^{-1}\, a_\varepsilon\, ds \to 0 \text{ uniformly in } [0, T],$$

but this follows from Lemma 3. Q.E.D.

Corollary 1 extends Theorem, p. 45 of [1] (conditions about partial derivatives of the data and condition (2.7a) of p. 45 may be omitted).

The following is an example of control system without approximability.

EXAMPLE 3. Consider

$$\dot{x}_1 = x_2 + u^{1/2}, \qquad \dot{x}_2 = -x_1 + u; \quad T = \tfrac{1}{17};$$
$$\mathscr{U} = [0, +\infty); \qquad s(x_1, x_2) = x_1.$$

Given $a_\varepsilon(t) = \sin(t/\varepsilon)$ and the initial conditions $x_1(0) = 0$, $x_2(0) = -2$, we obtain (notations of Definition 3) for all $t \in [0, T]$ and $\varepsilon > 0$,

$$x_{2\varepsilon}(t) = -2 + v_\varepsilon(t) - 2 \int_0^t x_{2\varepsilon}(s) \sin(s/\varepsilon)\, ds$$
$$+ \int_0^t \sin^2(s/\varepsilon)\, ds + \int_0^t x_{2\varepsilon}^2(s)\, ds,$$

where $v_\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly in $[0, T]$. Since $\int_0^t \sin^2(s/\varepsilon)\, ds \not\to 0$, we get $x_{2\varepsilon} \not\to x_{20}$, so contradicting approximability.

The following theorems exhibit classes of nonlinear control systems fulfilling the approximability property.

THEOREM 4. *The control system*

$$z^{(n)} = g(t, z, z', z'', \dots, z^{(n-1)}, u)$$

with $m = 1$, z *a scalar variable*, $u \in \mathscr{U}$, *verifies the approximability property under the following assumptions: g is a Carathéodory function, strictly monotone in u on* $\mathscr{U}$ *for every t and x near S*,

$$-\sum_{i=1}^{n-1} x_{i+1} \frac{\partial s}{\partial x_i} \in \frac{\partial s}{\partial x_n}\, g(t, x, U),$$
$$G \text{ is bounded and } \left| \frac{\partial s}{\partial x_n} \right| \geqslant c > 0 \text{ near } S; \tag{25}$$

$$\left(\frac{\partial s}{\partial x_n} \right)^{-1} \text{ and } G \text{ are locally Lipschitz near } S. \tag{26}$$

Proof. The control system is given in the form (1), (2), (3) with

$$f_j(t, x, u) = x_{j+1}, \qquad 1 \leqslant j \leqslant n-1,$$
$$f_n(t, x, u) = g(t, x, u).$$

Condition (7) holds by (25). Let V be a suitable neighborhood of S such that $g(t. x, \cdot)$ is one-to-one if $x \in V$. Given $p > 1$ and $M \in L^p(0, T)$ let a_ε in the corresponding class H.

Then setting

$$g_\varepsilon(t, x) = f[t, x, u^*(t, x, a_\varepsilon(t))], \qquad \varepsilon > 0,$$
$$g_0(t, x) = f[t, x, u^*(t, x, 0)],$$

we see that g_ε, $\varepsilon \geqslant 0$, verify (22) by (26), and (23) by (25). Since

$$g_\varepsilon(t, x) - g_0(t, x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \left(\dfrac{\partial s}{\partial x_n}\right)^{-1} a_\varepsilon(t) \end{pmatrix}$$

the conclusion follows from Lemma 2. Q.E.D.

THEOREM 5. *The control system* (1), (2), (3) *verifies the approximability property if*

$$f(t, x, u) = A(t, x) + B(t, x)\, h(u)$$

under the following assumptions: A, B are Carathéodory functions and h is continuous and one-to-one from $\mathscr{U}$ to R^m; $G(x)\, B(t, x)$ is nonsingular and

$$-[(GB)^{-1} GA](t, x) \in h(U) \qquad \textit{for every } t \textit{ and } x \textit{ near } S; \tag{27}$$

$A - B(GB)^{-1} GA$ is locally 1-integrably Lipschitz and of linear growth, $B(GB)^{-1}$ is locally q-integrably Lipschitz and of linear q-growth, $q > 1$; (28)

$$B(\cdot, x)[G(x)\, B(\cdot, x)]^{-1} \in L^q(0, T) \qquad \textit{if } x \textit{ is near } S. \tag{29}$$

Proof. Condition (7) follows from (27). Let $p > q/(q-1)$ and given M let $a_\varepsilon \in H$. Then (22) and (23) are fulfilled by (28) if

$$g_\varepsilon = A + B(GB)^{-1}(a_\varepsilon - GA),$$
$$g_0 = A - B(GB)^{-1} GA.$$

Since

$$g_\varepsilon(t, x) - g_0(t, x) = B(t, x)[G(x)\, B(t, x)]^{-1} a_\varepsilon(t)$$

the conclusion follows from Lemmas 2 and 3. Q.E.D.

Summarizing, we may define the equivalent control, obtain equivalence between sliding modes corresponding to the equivalent controls and states corresponding to discontinuous feedback laws in the sense of Filippov, and fulfil the approximability property for the following control systems (A), (B).

(A) $z^{(n)} = g(t, z, z', ..., z^{(n-1)}, u)$

with sliding surface

$$s[z, z', ..., z^{(n-1)}] = 0$$

and control constraint $u \in \mathscr{U}$, $m = 1$, under the following assumptions:

$s \in C^2(\Omega)$, $\partial s/\partial x$ bounded, $|\partial s/\partial x_n| \geqslant c > 0$ and $(\partial s/\partial x_n)^{-1}$ locally Lipschitz near S; g measurable in t, once continuously differentiable in x, twice in u; g and $\partial g/\partial x$ locally integrably bounded, $|\partial g/\partial u| \geqslant C > 0$ locally, both near S; U a closed interval such that for every x near S and t

$$0 \in \sum_{j=1}^{n-1} x_{j+1} \frac{\partial s}{\partial x_j} + \frac{\partial s}{\partial x_n} g(t, x, \mathscr{U}).$$

(B) $\dot{x} = A(t, x) + B(t, x)\, h(u)$

with sliding manifold (2), control constraint (3) under the following assumptions:

A, B are measurable in t, continuously differentiable in x, h is continuously differentiable and one-to-one on $\mathscr{U}$, $\mathscr{U}$ is closed and $h(\mathscr{U})$ is convex, GB and $GB(\partial h/\partial u)$ are nonsingular for every t, x near S and $u \in \mathscr{U}$, $B(\partial h/\partial u)$ is locally bounded near S, $(GB(\partial h/\partial u))^{-1}$ $(A + Bh)$, $(GB(\partial h/\partial u))^{-1}$ $((\partial A/\partial x) + (\partial B/\partial x)\, h)$ and $(\partial A/\partial x) + (\partial B/\partial x)\, h$ are locally integrably bounded, moreover conditions (8), (28), and (29) are verified; $s \in C^2(\Omega)$ and $-[(GB)^{-1}\, GA](t, x) \in h(U)$ for every t and x near S.

The statements in (A), (B), follows immediately from theorems 1, 2, 4, 5.

Remark. Simple modifications of the above results allow us to consider the more general case of the time-dependent sliding manifolds (as required, e.g., in [3]).

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